# Poiseuille Flow of Two Immiscible Fluids Between Flat Plates with Applications to Microfluidics

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Abstract

An analytic solution is derived for fully developed flow of two immiscible fluids between two flat plates when the fluids have arbitrary viscosity ratios and arbitrary flow rate ratios. This generalizes the solution in Bird, *et al.*[1] when the two fluids take up exactly the same space (i.e. the thickness of each fluid is one-half the total thickness). This solution is useful, for example, in certain microfluidic devices when two fluids flow in laminar contact; knowing the position of the dividing streamline is critically important to accurate modeling of diffusion across the interface. Since diffusion is often slow in comparison to the channel transit time, an approximate expression is provided to identify the limits of diffusion about the dividing streamline. Thus, it is possible to obtain *a priori* a good estimate of the concentration band as the fluids move down the device.

## 1. Analytical Solution

Fully developed flow of two immiscible fluids between two flat plates is solved in Bird, *et al.*[1] when the two fluids take up exactly the same space (i.e. the thickness of each fluid is one-half the total thickness). That solution is generalized here to allow any fraction, which ultimately will depend upon the flow rate ratio. The notation follows that of Bird, *et al.*[1] and the domain is illustrated in Figure 1. Let the total distance between the two flat plates be $H$, and the fraction filled with the lower fluid (I) be $f$. The domain stretches from $x = -f H$ to $x = (1-f) H$, i.e. the interface is at $x = 0$. The viscosities of the fluids are $\mu_1$ and $\mu_2$, and the flow rates are $Q_1$ and $Q_2$.

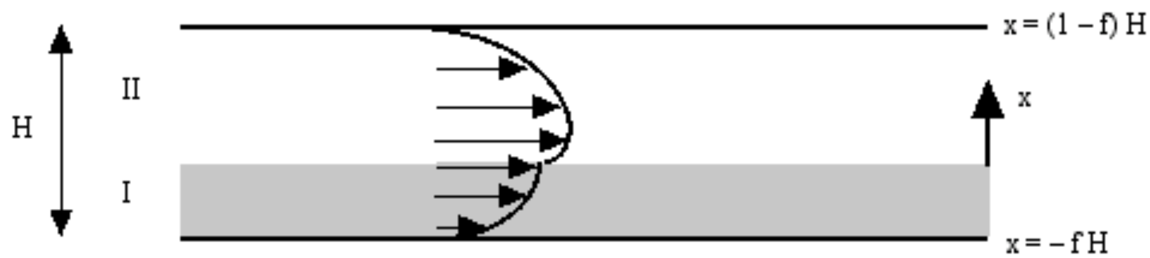


**Figure 1. Flow of two immiscible fluids between parallel plates**

The stress profiles are

$$\tau_1 = \frac{\Delta p}{L} x + C_1, \quad \tau_2 = \frac{\Delta p}{L} x + C_1$$

The constant $C_1$ is the same in both fluids since the stresses are the same at the interface, $x = 0$. For Newtonian fluids we have

$$\tau = -\mu \frac{dv}{dx}, \text{or } -\mu_1 \frac{dv_1}{dx} = \frac{\Delta p}{L} x + C_1, \text{ and } -\mu_2 \frac{dv_2}{dx} = \frac{\Delta p}{L} x + C_1$$

These equations can be integrated to give

$$v_1 = -\frac{\Delta p}{2\mu_1 L} x^2 - \frac{C_1}{\mu_1} x + C_2, \quad v_2 = -\frac{\Delta p}{2\mu_2 L} x^2 - \frac{C_1}{\mu_2} x + C_2$$

The boundary conditions require

$$v_1(-fH) = 0, \quad v_2[(1-f)H] = 0$$

Applying these conditions to the velocities and solving for $C_1$ and $C_2$ gives

$$C_1 = \frac{\Delta p H}{2L} \frac{\mu_2 f^2 - \mu_1 (1-f)^2}{f\mu_2 + (1-f)\mu_1}, \quad C_2 = \frac{\Delta p H^2}{2L} \frac{f(1-f)}{f\mu_2 + (1-f)\mu_1}$$

The flow rates are

$$Q_1 = \int_{-fH}^{0} v_1 dx, \quad Q_2 = \int_{0}^{(1-f)H} v_2 dx$$

$$Q_1 = -\frac{\Delta p}{6\mu_1 L} (fH)^3 + \frac{C_1}{2\mu_1} (fH)^2 + C_2 (fH)$$

$$Q_2 = -\frac{\Delta p H^3}{6\mu_2 L} (1-f)^3 - \frac{C_1 H^2}{2\mu_2} (1-f)^2 + C_2 H (1-f)$$

One can specify the fraction, $f$, and calculate the ratio of flow rates directly. When $f$ = 0.5 it gives the solution in Bird, *et al.* Alternatively, the flow rate ratio, $R = Q_1/Q_2$, can be specified, and the value of $f$ is determined to meet this ratio. When the two viscosities are equal, this should give the equation for a fully developed flow between two parallel plates, which it does. A MATLAB programs is used to solve for f when given $R = Q_1/Q_2$.

While this solution is for immiscible fluids, it may be a good approximation for miscible fluids in a microfluidic channel when the Reynolds number is small, which it usually is, since then no recirculation or instabilities will be present. One relevant situation is the laminar diffusion interface, which is established when two fluids of different chemical compositions are brought into laminar contact. Since they have different concentrations, diffusion takes place perpendicular to the interface as the fluids move down the channel. While the complete solution to the problem involves solving the convective diffusion equation, and including the possible effect of concentration on viscosity, there are situations in which an approximation is feasible. If one fluid contains a substance that increases its viscosity relative to the other fluid, but does not diffuse appreciably, then the viscosities of both fluids remain essentially constant as they move down the channel. Other chemicals, though, may diffusion more rapidly. For example,[2] if one fluid is serum and the other is water, one fluid (serum) contains proteins (albumin representing the most concentrated among them), that diffuses slowly, and metabolites (creatinine being one of many), which diffuses more rapidly. The proteins affect the viscosity, but the creatinine hardly does. In order to make use of such a channel to separate metabolites from proteins, one would like to know how far the albumin diffuses sideways as the flow proceeds down the channel, given the flow rates for serum and water. This modeling capability would then permit identification of flow rates for which the ratio of metabolite to protein is particularly high. This is still an approximation, since there is some axial diffusion, but it is a good approximation and can give guidance in the design and operation of devices.

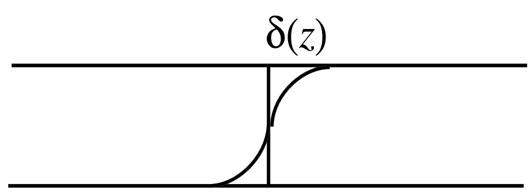


**Figure 2. Diffusion problem around dividing streamline**

Thus, consider the following problem for diffusion about the dividing streamline at $x = f$, as sketched in Figure 2. Now the coordinate $x$ is taken from 0 to H and the dividing streamline is at $x = f\,$H.

$$v(f)\frac{\partial c}{\partial z} = D\frac{\partial^2 c}{\partial x^2}, \frac{\partial c}{\partial x}(\pm\infty, z) = 0$$
$$c(x,0) = 1 \text{ for } x \le f\,H, = 0 \text{ for } x > f\,H$$

The velocity is the velocity at the dividing streamline; this assumption can be removed by using a numerical solution. With this assumption, the solution centered on $x = f\,$H is

$$c = 0.5\,erfc(\eta), \quad \eta \equiv \frac{x - f\,H}{\sqrt{4\dfrac{Dz}{v(f)}}}$$

The thickness of the diffusion layer is taken as the distance over which the concentration falls from 0.5 to 0.01 (2% criterion). This happens when the complementary error function is 0.01/0.5 = 0.02, or at an argument of $\eta$=1.645.

$$\delta = \sqrt{10.8\frac{Dz}{v(f)}}$$

This solution obviously cannot be used once the diffusion thickness reaches one of the walls. This is the thickness that the fluid diffuses from one fluid to another in a distance $z$ downstream. An approximate solution to this problem is[3]

$$c = 0.5(1-\eta^2), \eta = \frac{x - f\,H}{\delta}, \delta = \sqrt{12\frac{Dz}{v(f)}}$$

Either of these functions can be used to estimate the diffusion thickness above and below the dividing streamline under the stated assumptions. Other numerical experience has shown that these assumptions are quite good in realistic cases. In the case of albumin and creatinine, in a length of 2 cm with a velocity of 2 mm/sec, the creatinine diffusion length ($D$ = 9.2 $10^{-10}$ $m^2/s$) was 315 μm, but the albumin diffusion length ($D$ = 6.7 $10^{-11}$ $m^2/s$) was 85 μm. The microfluidic device thus provided one stream that had very little albumin in it provided it was obtained at a distance $f\,H + \delta$ above the bottom.

2. Numerical Results

Figure 3 shows the dimensionless velocity profiles when the flow rate ratios vary from 0.2 to 5 and the viscosity ratios is 5. The coordinate $x$ is dimensionless, scaled by the height, $H$, with the dividing streamline at $x$ = 0. The velocity is scaled such that $\Delta p H^2/2L = 1$. As expected, when the viscosities are different, the slopes of the velocity profiles are different at the dividing streamline. Of more interest is the difference between the flow rate ratios and the physical location of the actual streamline. Figure 4 shows the difference between the $f$ predicted by the analysis and the value of $f$ that would be predicted based upon the ratio of flow rates, $Q_1/(Q_1+Q_2)$. As can be seen, there is considerable difference in those values when the flow rate ratio is small, regardless of the viscosity ratio, up to 100% difference in some cases. Thus, the ratio of flow rates does not give a good estimate of the location of the dividing streamline.

Consider next a numerical solution of the diffusion problem given above with $v(f)$ replaced by the actual velocity, scaled so that the streamline velocity is 2 mm/sec. The finite difference method[4] is used to solve the convective diffusion equation when the height is 380 μm, the diffusivities are as given, the viscosity ratio is 5 and the flow ratio is 1.0. In this case the $f$ = 0.5735. Figure 5 shows the numerical solution at the exit using $x$ scaled by $H$ and with the dividing streamline at $x = f$. The velocity profile has a dramatic effect on the rapidly diffusing species (creatinine) but almost no effect on the albumin because it diffuses so slowly. If one takes the diffusion layer as the distance between the streamline value and the point where $c$ = 0.01 or 0.99, the diffusion layer for albumin is 85 μm on one side and 95 μm on the other side. These numbers are comparable to the estimates given above but show the effect of the velocity profile on each side of the dividing streamline.

Simulations for two-dimensional and three-dimensional configurations, including entry and exit geometries, are available elsewhere.[5]

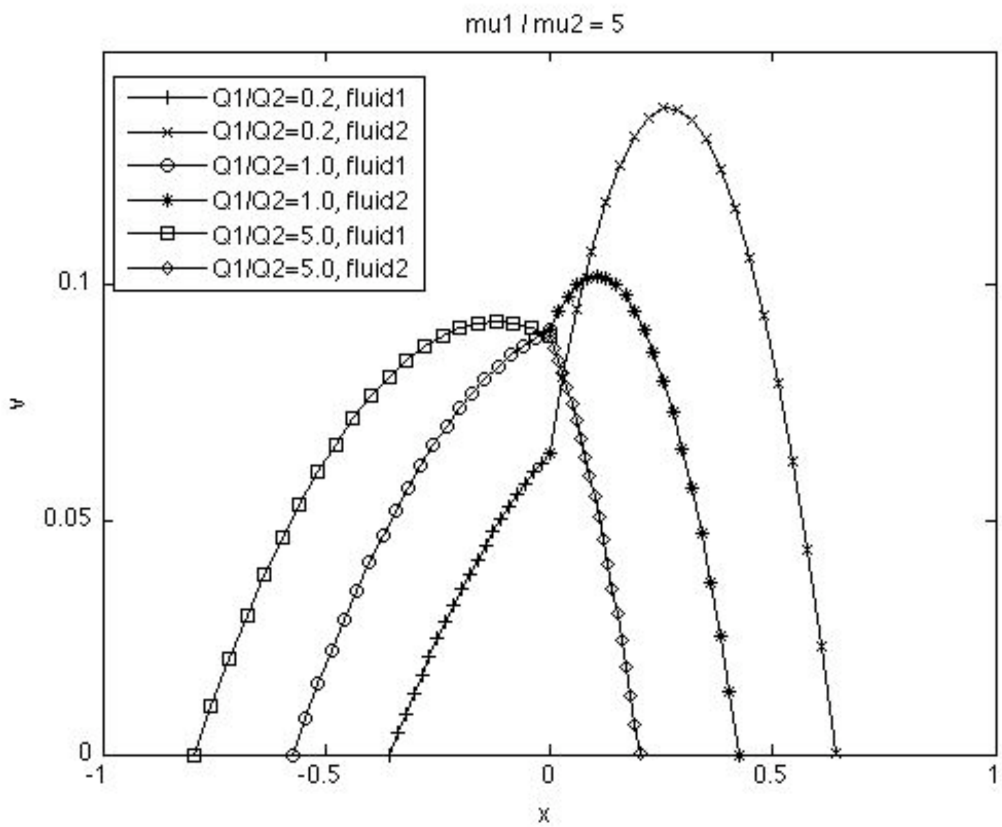


**Figure 3. Velocity profile for two immiscible fluids flow when $\mu_1/\mu_2$=5; all curves are for the same pressure drop per length**

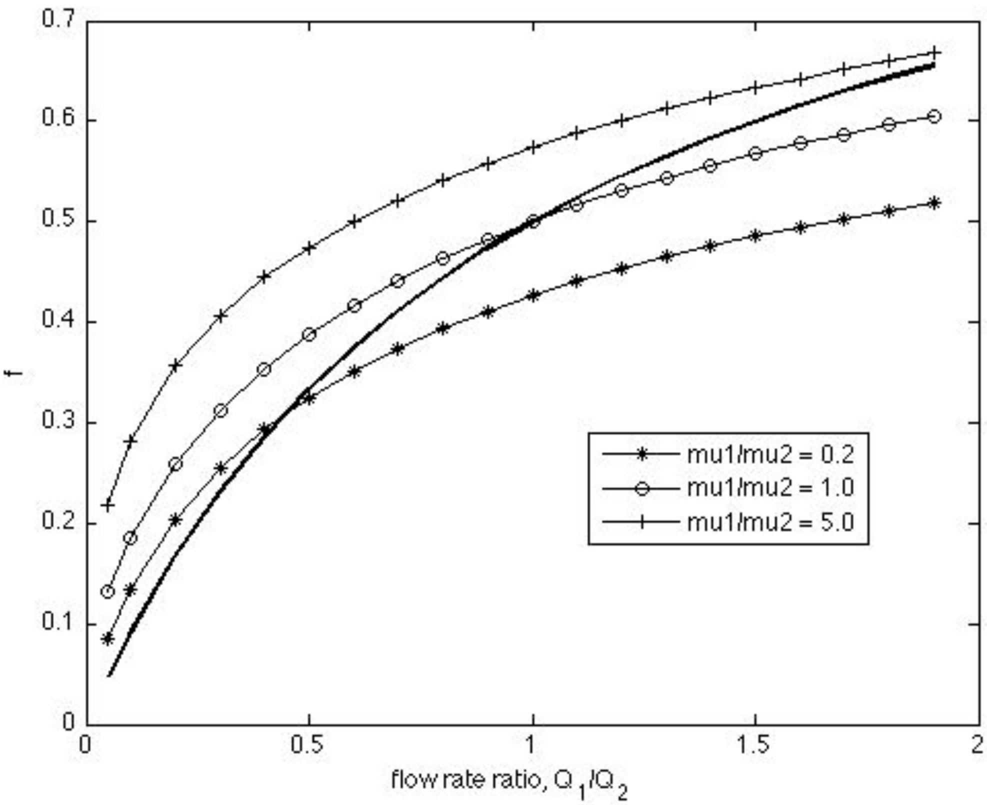


**Figure 4. Fraction of the height showing location of dividing streamline; solid line is the ratio of flow rates, $Q_1/(Q_1+Q_2)$**

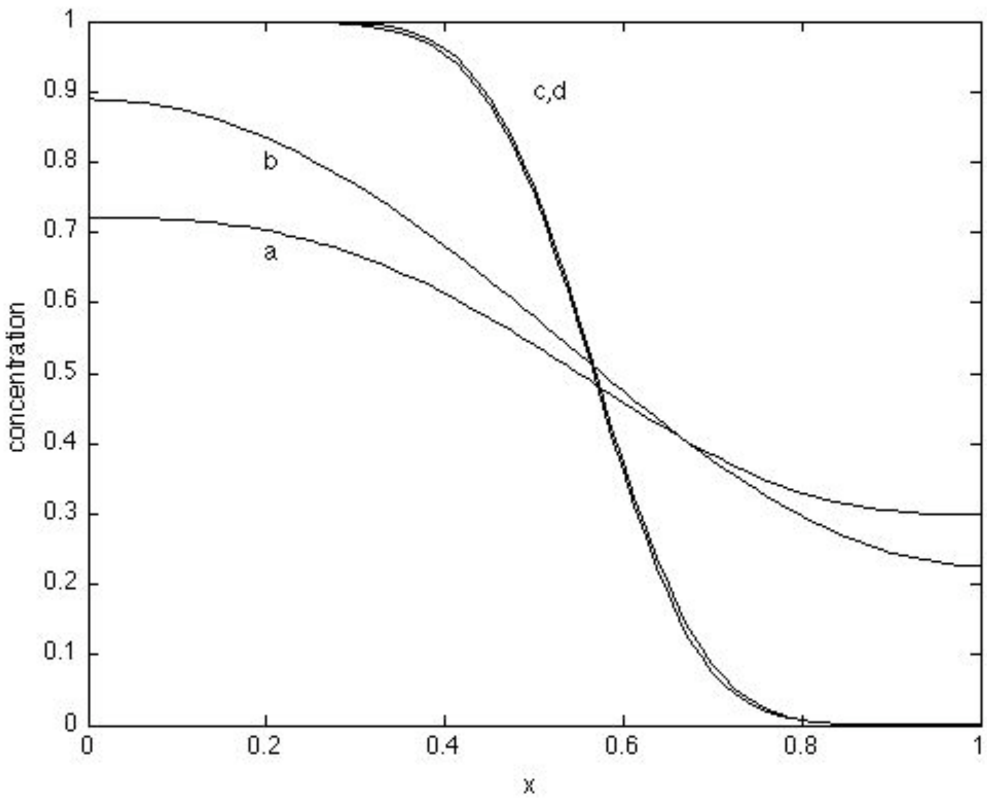


**Figure 5. Numerical solution at the exit; viscosity ratio = 5, flow rate ratio = 1; (a, b) creatinine with a velocity profile (a) and constant velocity (b); (c, d) albumin with a velocity profile and constant velocity**

## 3. Conclusions

An analytical solution is derived for the flow of two immiscible fluids between two flat plates. It is valid for any ratio of viscosities and any ratio of flow rates, thus generalizing the results of Bird, *et al.*[1] When combined with an analytical solution of diffusion in the transverse direction, it can be used to estimate the thickness of diffusion layers when the two fluids have different chemical compositions.

## 4. References

1. Bird, R. B., W. E. Stewart, E. N. Lightfoot, *Transport Phenomena*, 2nd ed., Wiley (2002), pp. 56-68.

2. Shaw, A., B. Schattka, M. Alexander, S. Low-Ying, A. Man, *Metabolic fingerprinting of biofluids by infrared spectroscopy: Modeling and optimization of flow rates for laminar fluid diffusion interface sample preconditioning,* submitted.

3. Finlayson, B. A., *The Method of Weighted Residuals and Variational Principles*, Academic Press (1972), p. 45.

4. Finlayson, B. A., *Nonlinear Analysis in Chemical Engineering,* McGraw-Hill (1980).

5. Finlayson, B. A., R. A. Shaw, *Modeling Microfluidic Separations using Comsol Multiphysics,* Comsol Conference 2010, Boston.

## 5. Acknowledgment

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