The following problem accompanies the book, Method of Weighted Residuals and Variational Principles, by Bruce A. Finlayson, a SIAM Classic reprinted in 2014. The original version was printed by Academic Press in 1972. See www.ChemEComp.com/MWR. Order the book from the Society of Industrial and Applied Mathematics, www.SIAM.org. The problems and solutions refer to equations and references in that book.

Problem 1. Solve the following problem [a simplification of Eq. (11.50)] using an expansion in sine functions. Apply Eq. (11.54) to calculate an upper limit to the pointwise error as a function of *n*. Determine the maximum actual pointwise error. How does this compare with the result from Eq. (11.54)? How does it change with *n*? Calculate the variational integral with the sine functions, as a function of *n*.

The problem is:

$$\frac{d^2y}{dx^2} = -1, \quad y(0) = y(1) = 0 \tag{1.1}$$

This problem, simple as it is, is the equation for flow of a Newtonian fluid between two flat plates.

$$\mu\frac{d^2u}{dz^2} = -\frac{\Delta p}{L}, \quad u(0) = u(2H) = 0 \tag{1.2}$$

with

$$u' = \frac{\mu u H^2}{\Delta p / L}, \quad z' = \frac{z}{H}$$

Eq. (1.2) becomes Eq. (1.1). ($u' \to y, \quad z' \to x$).

Part a. You can easily solve the equation to get the exact solution (in polynomials), but do it by minimizing the variational integral when the trial function is a polynomial that satisfies the boundary conditions. Also derive the Euler equation and second variation.

Part b. Use the sine functions and minimize the variational integral.

Part c. Use the sine functions and apply collocation.

Part d. Use the sine functions and apply the least squares method.

Comment. Eq. (11.54) is given by Kantorovich and Krylov when $q > 0$. Here $q = 0$. Does it still apply? If you take q as a small positive constant (10^{-20}) the theorem applies, and the error bounds are not affected within machine precision.