The following problem accompanies the book, Method of Weighted Residuals and Variational Principles, by Bruce A. Finlayson, a SIAM Classic reprinted in 2014. The original version was printed by Academic Press in 1972. See www.ChemEComp.com/MWR. Order the book from the Society of Industrial and Applied Mathematics, www.SIAM.org. The problems and solutions refer to equations and references in that book.

Problem 10 Solve the following elliptic partial differential equation, which governs laminar flow in a duct (§4.1 and §7.3) among other phenomenon.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-1 \text{ in } A, \text{ with boundary condition } u=0 \text{ on } \mathrm{C}=\partial \mathrm{A} \qquad (7.62a, b)$$

Provide point-wise error bounds using the Green's function (p. 390). The mean-square bound is

$$\left\|u-u_n\right\|_2 \leq \frac{\left\|R_n\right\|_2\left\|L^{-1}\right\|_2}{1-K\left\|L^{-1}\right\|_2} \qquad (11.174)$$

To obtain the point-wise error bound from this, see Problems 7-9.

Part a. Consider first the one-dimensional problem

$$\frac{d^2 u}{dx^2}=-1, u(0)=u(1)=0$$

Develop the mean-square error bounds and the point-wise error bounds, using the information in Problem 7. Since the exact solution is a quadratic, the first approximation that has piecewise continuous first derivatives (finite difference method with one interior point) gives the exact solution at x = ½. However, evaluate the constants for the error bounds.

Part b. Develop the same error bounds, in terms of the Green's function, for the 2D problem, (7.62a,b). Find an error bound for the flow rate, *i.e.* u integrated over A.

Part c. Solve Eq. (7.62a,b) on a square, $0 \le x \le 1, 0 \le y \le 1$ with the finite element method and compare the error bound for flow rate with the exact value in Table 4.1 and the upper and lower bounds in Eq. (7.76).

Part d. Solve in a rectangle with a 2:1 aspect ratio, as in Problem 6, $0 \le x \le 1/2, 0 \le y \le 1$. Compare the bounded error with the estimated error found in Problem 6.

Hint: While the point-wise error bounds required u = 0 on the boundary, the problem is symmetric about the origin and can be solved in one-fourth of the domain.

Background.

One dimensional problem. First consider the one-dimensional problem:

$$\frac{d^2u}{dx^2} = -1, \quad u(0) = u(1) = 0$$

The solution is $u = \frac{1}{2}(x - x^2)$.

The Green's function for this equation is given in Table 1 of Ferguson and Finlayson and by Courant and Hilbert, p. 371.

$$G(x;\xi) = \begin{cases} (1-x)\xi & \text{if} \quad \xi < x \\ (1-\xi)x & \text{if} \quad x \le \xi \end{cases}$$

The error bounds are provided by Th. 1 and 2 in Ferguson and Finlayson and are summarized in Problems 7-9.

$$\|u - u_n\| \le \frac{\|R_n\|\|L^{-1}\|}{1 - K\|L^{-1}\|} \qquad (11.174)$$

$$\|L^{-1}\| \equiv \left\{ \int_0^1 \int_0^1 G^2(x,\xi) x^{a-1} dx\, d\xi \right\}^{1/2}$$

$$K_2^2 = \max_{0 \le x \le 1} \int_0^1 G^2(x;\xi) d\xi$$

$$\|u - u_n\|_\infty \le K_2 \left[K \|u - u_n\|_2 + \|R\|_2 \right]$$

The Lipshitz constant, *K*, for *f*(*x*) = –1, is zero. The other expressions simplify to

$$\|u - u_n\| \le \|R_n\|\|L^{-1}\|, \quad \|u - u_n\|_\infty \le K_2 \|R\|_2$$

The following result is also a Green's function for the 1D problem. It is simplified from the Green's function for the 2D problem, given below.

$$G(x;\xi) = \frac{2}{a\pi^2} \sum_{k=1}^{\infty} \frac{\sin(\frac{k\pi x}{a})\sin(\frac{k\pi \xi}{a})}{\frac{k^2}{a^2}}$$

In the one-dimensional case, this function is the same as given above. This is most easily seen by plotting them both or calculating

$$u(x) = \int_0^1 G(x,\xi) f(\xi)\, d\xi$$

and comparing it to $u = \frac{1}{2}(x - x^2)$. Here $f(x)=-1$. The two functions can be plotted, too, and are seen to be the same.

Two-dimensional problem. Next consider the two-dimensional problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1, \quad 0 \le x \le a, \quad 0 \le y \le b, \quad u = 0 \text{ the boundaries.}$$

Courant and Hilbert, p. 379, give a Green's function for this problem and the 3D problem. They note that it is not uniformly convergent, but Piosky, p. 362, says that it is convergent except at the points $(x, y) = (\xi, \eta)$.

$$G(x,y;\xi,\eta) = \frac{4}{ab\pi^2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(\frac{k\pi x}{a})\sin(\frac{k\pi\xi}{a})\sin(\frac{m\pi y}{b})\sin(\frac{m\pi\eta}{b})}{\frac{k^2}{a^2} + \frac{m^2}{b^2}}$$

In the two dimensional case, the series does not converge uniformly and is difficult to use. An alternative is provided by Kevin D. Cole in the Green's Function Library at the University of Nebraska-Lincoln: http://www.engr.unl.edu/~glibrary/glibcontent/node24.html.
In the notation here, it is

$$G(x,y;\xi,\eta) = \sum_{k=1}^{\infty} \frac{\sin(\frac{k\pi y}{b})\sin(\frac{k\pi\eta}{b})}{(b/2)} P_k(x,\xi)$$

$$A = \exp\left[-\frac{k\pi}{b}(2a - |x-\xi|)\right] - \exp\left[-\frac{k\pi}{b}(2a - x - \xi)\right] +$$
$$+\exp\left[-\frac{k\pi}{b}|x-\xi|\right] - \exp\left[-\frac{k\pi}{b}(x+\xi)\right]$$

$$P_k(x,\xi) = A \Big/ \left\{\frac{2k\pi}{b}\left[1 - \exp\left(-\frac{2k\pi a}{b}\right)\right]\right\}$$

Just as in the one-dimensional case, a bound can be obtained in terms of the Green's function squared. For the problem

$$\nabla^2 u = -f, \quad \text{with} \quad u = 0 \text{ on } \partial\Omega$$

Mikhlin (p. 97) derives the following bound:

$$u_n - u = \int_\Omega G(x,y;\xi,\eta)(\nabla^2 u_n + f)(\xi,\eta)d\xi d\eta$$

$$\left|u_n(x,y) - u(x,y)\right| \le \sqrt{\left(\int_\Omega G^2(x,y;\xi,\eta)\,d\xi\,d\eta\right)} \text{ X } \sqrt{\left(\int_\Omega (\nabla^2 u_n + f)^2\,d\xi\,d\eta\right)}$$

He points out that the Green's function satisfies the inequality

$$0 \le G(x,y;\xi,\eta) \le \left|\ln r\right| + b\,,$$

where a and b are constants, so that the integral of the Green's function is bounded by a constant, C.

$$\int_\Omega G^2(x,y;\xi,\eta)\,d\xi\,d\eta < C$$

Here the constants (evaluated using the Cole form of the Green's function) is $C = K_2^2$ = .00291 for *a* = 1, *b* = 1, and 0.0166 for *a* =1, *b* = 2.

References

Courant, R. and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience, New York, NY, 1953.

Pinsky, M. A., *Partial Differential Equations and Boundary-value Problems with Applications*, McGraw-Hill, New York, NY, 1991.

Green's Function Library at the University of Nebraska-Lincoln, Kevin D. Cole: http://www.engr.unl.edu/~glibrary/glibcontent/node24.html, accessed Oct. 17, 2014.

Mikhlin, S. G., *Variational Methods in Mathematical Physics*, Macmilllan Co., New York, 1964; translated by T. Boddington, introduction by L. I. G. Chambers.