The following problem accompanies the book, Method of Weighted Residuals and Variational Principles, by Bruce A. Finlayson, a SIAM Classic reprinted in 2014. The original version was printed by Academic Press in 1972. See www.ChemEComp.com/MWR. Order the book from the Society of Industrial and Applied Mathematics, www.SIAM.org. The problems and solutions refer to equations and references in that book.

Problem 2. The Theorem, Eq. (11.124), is proved by Kantorovich and Krylov for *p(x)* = 1 and *q(x)* > 0, as is clear from the bounds, when the trial functions are sine functions. They prove a different theorem (K&K, p. 337) when the trial functions are polynomials, $\varphi_i(x) = x^i(1-x), i = 1,2,\ldots,n$.

$$\left|\frac{\lambda_k^{(n)} - \lambda_k}{\lambda_k^{(n)}}\right| < \frac{N\left|\lambda_k^{(n)}\right|}{n^2(n+1)^2(n+2)(n-1)}$$

$$N = \left\{\max\left|\frac{q''}{q^{1/2}}\right| + 2\max|q'|\sqrt{\left|\lambda_k^{(n)}\right|\frac{\max q^{1/2}}{\min q^{1/2}}} + \max q^{3/2}\left|\lambda_k^{(n)}\right|\right\}^2$$

Solve the problem with q = 1.

$$\frac{d^2y}{dx^2} + \lambda y = 0, y(0) = y(1) = 0$$

Part a. Find the exact solution.

For all parts compare the eigenvalues with the exact values. Compare the error bound (if it applies) with the actual error. The methods given below can easily be applied to more complicated eigenvalue problems, where the *p* and *q* take other values, but these may not satisfy the conditions required by the theorem in Kantorovich and Krylov.

Part b. Compute the first two approximate eigenvalues with the polynomial expansion.

Depending upon the resources available to you:

Part c. Use MATLAB to obtain higher eigenvalues as in (b).

Part d. Use MATLAB with …

Part e. Solve using Comsol Multiphysics. To solve this, choose the 'Coefficient Form PDE' and 1D. Set the'Geometry' as a line from 0 to 1. Be sure the discretization option is displayed (click the menus at the top). The trial functions will be finite elements. You can choose the mesh size as well as the discretization. Try three options: Lagrange, linear; Lagrange, quadratic, and Hermite, cubic. The Lagrange, linear does not satisfy the condition that the trial function be in C^1. In 'Study' choose the number of eigenvalues to calculate. The number is restricted by the number of degrees of freedom in the trial function. (More information about Comsol Mulitphysics can be obtained at www.ChemEComp.com.)

Part f. Compare the solutions from Parts c, d, and e.