The following problem accompanies the book, Method of Weighted Residuals and Variational Principles, by Bruce A. Finlayson, a SIAM Classic reprinted in 2014. The original version was printed by Academic Press in 1972. See www.ChemEComp.com/MWR. Order the book from the Society of Industrial and Applied Mathematics, www.SIAM.org. The problems and solutions refer to equations and references in that book.

Problem 6 Solve the following elliptic partial differential equation, which governs laminar flow in a duct (§4.1 and §7.3) among other phenomenon.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-1 \text{ in } A\text{, with boundary condition } u=0 \text{ on } C=\partial A \qquad (7.62a, b)$$

The variational integral is Eq. (7.63).

$$I(u)=\int_A\left[-\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial y}\right)^2+2u\right]dA$$

For this problem, take the domain as a rectangle, $0\le x\le 1, 0\le y\le 2$.

As discussed on pages 358-9, Mikhlin proved that there is only one solution that converges in energy and in the mean to the solution. Trial functions that are complete in energy and linearly independent constitute a minimizing sequence. Kantorovich and Krylov show that polynomials in *x* and *y* satisfy this condition. They also suggest that one way to insure that the trial functions satisfy the boundary condition is to define the trial function as

$$u_n=w(x,y)\sum_i f_i(x,y)$$

where w = 0 on the boundary, *C.*

Part a. Take the first variation and find the Euler equation. Then find the second variation to determine if the variational method is a maximum principle. Relate the variational integral to the flow rate, Eq. 7.65. Will a flow rate determined from the variational principle be above or below the exact value?

Part b. The problem should be symmetric about x = ½ and y = 1. Thus, it makes sense to solve it only on a quarter of the space. Do this on the upper quarter of the domain, but with coordinates $0 \le x \le 0.5$ and $0 \le y \le 1$, and use only even powers of *x* and *y*. Solve the problem using the trial function:

$$u(x,y)=\left(\frac{1}{4}-x^2\right)\left(1-y^2\right)\sum_{i=1}^{nx}\sum_{j=1}^{ny}a_{(i-1)*ny+j}x^{2i-2}y^{2j-2}$$

Take as many terms as your patience allows, but definitely do $nx = ny = 1$ and $nx = 1$, $ny = 2$. Does the flow rate increase when an additional term is added? Compare with the friction factor for a rectangle in Table 4.2.

Part c. Using any program that allows continuous polynomials with continuous first derivatives, derive the solution for higher approximations. Show the variational integral as a function of the number of terms. While this does not provide error bounds (see the discussion on page 360), the rate of convergence is useful. Examine Schultz (1969c) to see if the theorems there can provide an error bounds.