The following problem accompanies the book, Method of Weighted Residuals and Variational Principles, by Bruce A. Finlayson, a SIAM Classic reprinted in 2014. The original version was printed by Academic Press in 1972. See www.ChemEComp.com/MWR. Order the book from the Society of Industrial and Applied Mathematics, www.SIAM.org. The problems and solutions refer to equations and references in that book.

Problem 1. Part a. The solution will be a quadratic polynomial in *x*, so use

$$y = a_0 + a_1 x + a_2 x^2$$

Applying the boundary conditions gives $a_0 = 0, \quad a_2 = -a_1$. Thus, the function is

$$y = a_1 (x - x^2)$$

The variational integral is

$$I[y] = \int_0^1 \left[\left(\frac{dy}{dx} \right)^2 - 2y \right] dx$$

And for the quadratic polynomial it is

$$I(y) = \int_0^1 a_1^2 (1-2x)^2 \, dx - \int_0^1 2 a_1 x (1-x) dx$$

Minimizing with respect to a_1 gives a_1= 0.5. For that value of a_1 the variational integral is −1/12. Any approximate solution should give a larger value than this.

The first variation gives

$$\delta I = \int_0^1 \left[2 \left(\frac{dy}{dx} \right) \left(\frac{d\delta y}{dx} \right) - 2\delta y \right] dx = \int_0^1 \left[-2 \left(\frac{d^2 y}{dx^2} \right) \delta y - 2\delta y \right] dx + 2 \left[\delta y \frac{dy}{dx} \right]_0^1$$

Since thre is no variation at the boundaries, the Euler equation and second variation are

$$\frac{d^2 y}{dx^2} + 1 = 0 \quad , \quad \delta^2 I = 2 \int_0^1 \left(\frac{d\delta y}{dx} \right)^2 dx$$

which is always positive, indicating a minimum principle.

Part b. The solution is expanded in sine functions, since the cosine functions would not be zero at $x = 0$. The following equations are used in parts b and c.

$$y_n = \sum_{j=1}^{n} \sin(j\pi x), \qquad \frac{dy_n}{dx} = \sum_{j=1}^{n} j\pi \cos(j\pi x), \qquad \frac{d^2 y_n}{dx^2} = -\sum_{j=1}^{n} (j\pi)^2 \sin(j\pi x)$$

The variational integral is

$$I[y_n] = \int_0^1 \left[\sum_{j=1}^{n} j\pi a_j \cos(j\pi x) \sum_{k=1}^{n} k\pi a_k \cos(k\pi x) - 2\sum_{j=1}^{n} a_j \sin(j\pi x) \right] dx$$

The mean square error is

$$\text{Mean Square Error } = \left\{ \int_0^1 \left[y_n - y_{exact} \right]^2 dx \right\}^{1/2}$$

The residual is

$$\text{Residual} = -\sum_{j=1}^{n} (j\pi)^2 \sin(j\pi x) \ + 1$$

and the mean square residual is

$$\text{Mean Square Residual } = \left\{ \int_0^1 \left[-\sum_{j=1}^{n} (j\pi)^2 \sin(j\pi x) \ + 1 \right]^2 dx \right\}^{1/2}$$

The variational integral is differentiated with respect to a_i to give the equations for a_i.

$$\sum_{i=1}^{n} A_{ki} a_i = f_k, \quad \text{where } A_{ki} = 2(k\pi)(i\pi) \int_0^1 \cos(i\pi x)\cos(k\pi x)\, dx = (k\pi)^2 \delta_{ki}$$

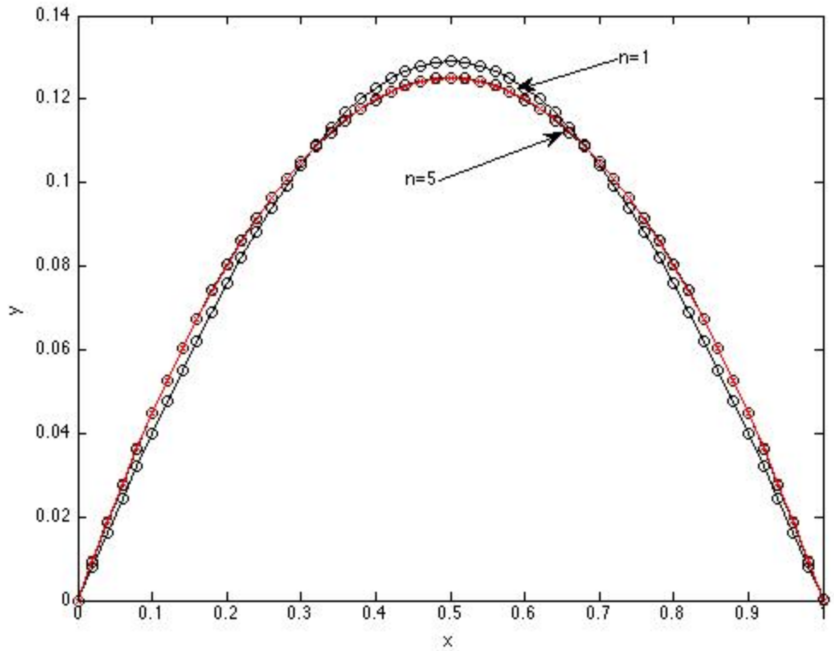
$$f_k = 2\int_0^1 \sin(k\pi x)\, dx = \frac{4}{k\pi} \text{ for } k \text{ odd, } 0 \text{ for } k \text{ even}$$

$$a_i = \frac{4}{(i\pi)^3} \text{ for } i \text{ odd, } 0 \text{ for } i \text{ even}$$

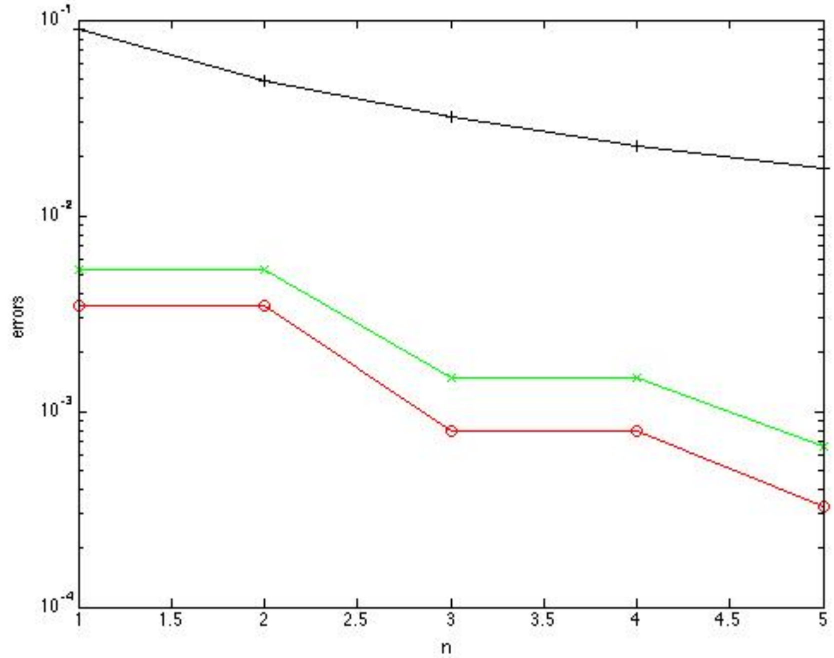
The results are shown below.

n	$I(y_n)$	$I(y_n)$- $I(y_{exact})$	Mean Square Error	Maximum Error (51 points)	Error Bound, Eq. (11.54)
1	−0.0821	0.0012	0.00345	0.0053	0.0898
2	−0.0821	0.0012	0.00345	0.0053	0.0489
3	−0.0831	1.92e-4	7.91e-4	0.00147	0.0317
4	−0.0831	1.92e-4	7.91e-4	0.00147	0.0227
5	−0.0833	6.01e-5	3.26e-4	6.66e-4	0.0173

As can be seen, the error bound is above the maximum error, but not too much above it, and it decreases with *n*, insuring a good approximation. The mean square error is much smaller, since the error is small over much of the domain. Shown below is the solution and an error plot. Note that the solutions with even *n* are not needed.



Solution using variational principle;
red is exact solution, black is collocation solution



Errors when using the variational principle

Part c. Collocation is applied with a uniform distribution of collocation points. Other choices are possible, of course. The collocation points are taken at

$$x_k = -\frac{1}{2n} + \frac{k}{n}$$

Evaluating the residual at these points gives

$$\sum_{j=1}^{n} A_{kj} a_j = f_k, \quad \text{where } A_{kj} = j^2 \sin(j\pi x_k), \quad \text{and } f_k = 1/\pi^2.$$

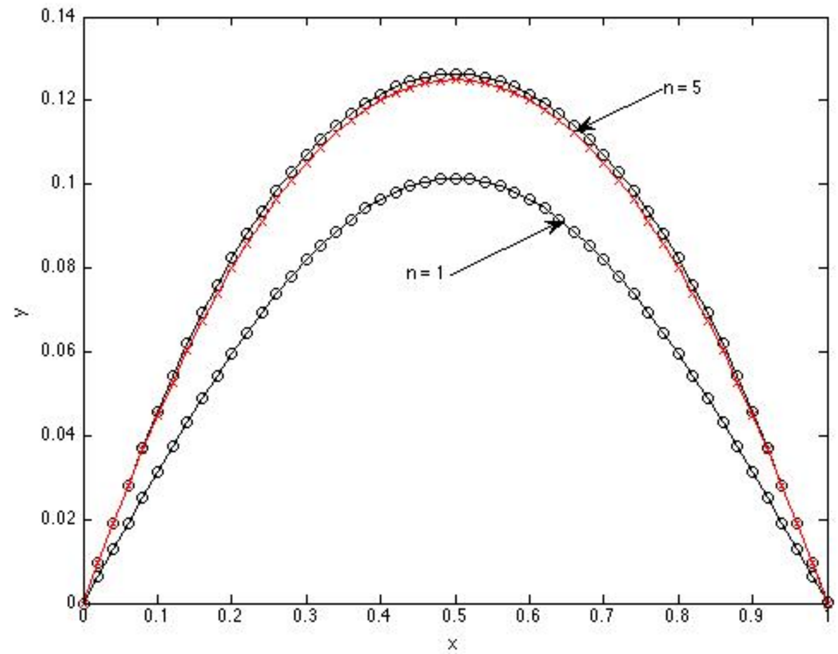
This set of linear equations is solved in MATLAB. The solution at the collocation points is

$$y_n(x_k) = \sum_{j=1}^{n} B_{kj} a_j, \quad \text{where } B_{kj} = \sin(j\pi x_k)$$

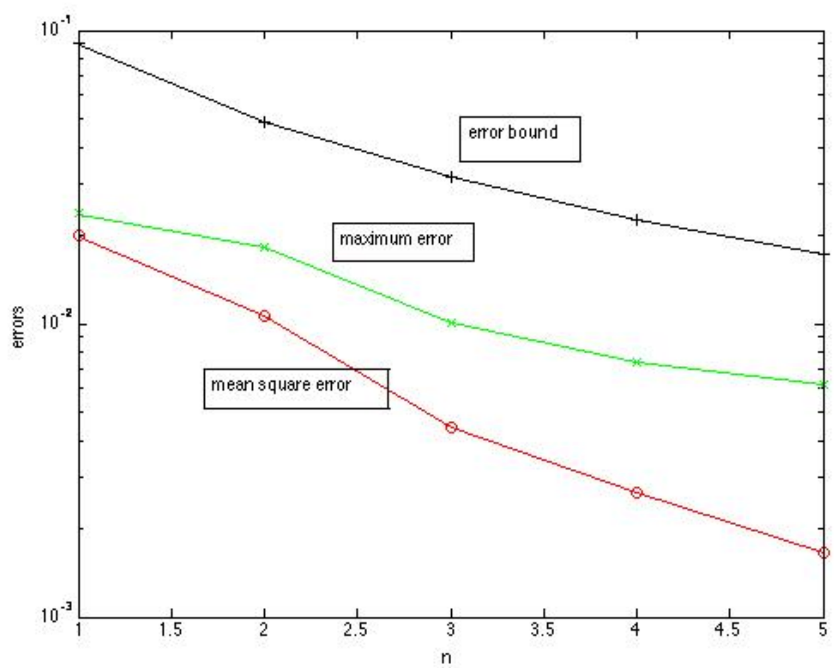
The results are shown below.

n	$I(y_n)$	$I(y_n) - I(y_{exact})$	Mean Square Error	Maximum Error (51 points)	Error Bound, Eq. (11.54)
1	−0.0783	0.005	0.01988	0.0237	0.0898
2	−0.0811	0.0022	0.01067	0.0183	0.0489
3	−0.0829	4.21e-4	0.004455	0.00634	0.0317
4	−0.0830	3.25e-4	0.002668	0.00418	0.0227
5	−0.0832	1.17e-4	0.001648	0.00242	0.0173

As can be seen, the error bound is above the maximum error, but not too much above it, and it decreases with *n*, insuring a good approximation. The mean square error is much smaller, since the error is small over much of the domain. Shown below is the solution and an error plot.



Solution using Collocation; red is exact solution, black is collocation solution



Errors when using the collocation method

Part d. $$\sum_{i=1}^{n} A_{ki} a_i = f_k, \quad \text{where } A_{ki} = 2k^2 i^2 \pi^4 \int_0^1 \sin(i\pi x)\sin(k\pi x)\,dx = (k\pi)^4 \delta_{ki}$$

$$f_k = 2k^2\pi^2 \int_0^1 \sin(i\pi x)\,dx = 4k\pi \text{ for } k \text{ odd, 0 for } k \text{ even}$$

$$a_i = \frac{4}{(i\pi)^3} \text{ for } i \text{ odd, 0 for } i \text{ even}$$

Thus, the solution for the least squares method is the same as for the variational principle, in part b. This happens in this case because the trial functions (sines) are orthogonal, as are the first and second derivatives (cosines and sines).