The following problem accompanies the book, Method of Weighted Residuals and Variational Principles, by Bruce A. Finlayson, a SIAM Classic reprinted in 2014. The original version was printed by Academic Press in 1972. See www.ChemEComp.com/MWR. Order the book from the Society of Industrial and Applied Mathematics, www.SIAM.org. The problems and solutions refer to equations and references in that book.

Problem 2.

Part a. Expand the solution in sines and cosines.

$$y = \sum_{n=1,2,..} a_n \sin(n\pi x)$$

Substituting into the differential equations gives

$$y'' + \lambda y = -\sum_{n=1,2,...} (n\pi)^2 a_n \sin(n\pi x) + \lambda \sum_{n=1,2,...} a_n \sin(n\pi x) = 0$$
$$\lambda_n = (n\pi)^2$$

The first two eigenvalues are 9.869605 and 39.4784.

Part b. The trial function is

$$y_n = (1-x)\sum_{i=1}^{n} a_i x^i = \sum_{i=1}^{n} a_i x^i - \sum_{i=1}^{n} a_i x^{i+1}$$
$$y_n' = \sum_{i=1}^{n} i a_i x^{i-1} - \sum_{i=1}^{n} (i+1) a_i x^i$$
$$y_n'' = \sum_{i=2}^{n} i(i-1) a_i x^{i-2} - \sum_{i=1}^{n} (i+1) i a_i x^{i-1}$$

The variational formulation is

$$\int_0^1 (1-x)x^j (y'' + \lambda y)dx = \sum_{i=2}^{n} i(i-1)a_i \int_0^1 (1-x)x^{j+i-2}\,dx - \sum_{i=1}^{n}(i+1)ia_i \int_0^1 (1-x)x^{j+i-1}dx$$
$$+ \lambda \sum_{i=1}^{n} a_i \int_0^1 (1-x)^2 x^{j+i} dx = 0$$

For the first approximation

$$-2a_1 \int_0^1 (1-x)x\,dx + \lambda a_1 \int_0^1 (1-x)^2 x^2 dx = 0, \text{ or } \lambda = \frac{2/6}{1/30} = 10$$

This is close to $\pi^2 = 9.869605$, the exact value of the first eigenvalue. The theorem does not apply to this first approximation due to the (n–1) in the denominator.

Higher approximations can be calculated from

$$\sum A_{ji}a_i + \lambda \sum B_{ji}a_i = 0$$

where

$$A_{ji} = i(i-1)\int_0^1 (1-x)x^{j+i-2}\,dx - i(i+1)\int_0^1 (1-x)x^{j+i-1}dx$$

$$B_{ji} = \int_0^1 (1-x)^2 x^{j+i}dx$$

The second approximation gives

$$\begin{bmatrix} -\frac{1}{3}+\lambda\frac{1}{30} & \frac{1}{3}-\frac{1}{2}+\lambda\frac{1}{60} \\ \frac{1}{3}-\frac{1}{2}+\lambda\frac{1}{60} & \frac{1}{6}-\frac{3}{10}+\lambda\left(\frac{1}{5}-\frac{2}{6}+\frac{1}{7}\right) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions are λ_1 = 10 = and λ_2 = 42. Both of these are above the exact values, but the first eigenvalue is closer to its exact value. This is characteristic of variational solutions, since the first eigenvalue is the minimization of a function, and the second eigenvalue is the minimization of the functional among functions orthogonal to the first one. Since the first one is only approximated, the error will be higher. The second approximation to the first eigenvalue is not improved over the first approximation because the second term in the trial function is not a symmetric function about x = 1/2.

For n = 2, the error bound is

$$\left|\frac{\lambda_k^{(n)} - \lambda_k}{\lambda_k^{(n)}}\right| < \frac{N\left|\lambda_k^{(n)}\right|}{n^2(n+1)^2(n+2)(n-1)},\ N = \left\{\left|\lambda_k^{(n)}\right|\right\}^2$$

$$\left|\frac{\lambda_k^{(n)} - \lambda_k}{\lambda_k^{(n)}}\right| < \frac{\left|\lambda_k^{(n)}\right|^3}{4(3)^2 4}$$

This gives a relative bound of 6.9 for the first eigenvalue and 515 for the second eigenvalue. Clearly they are much closer than that.

Part c. Using the formulas in Part b, it is necessary to evaluate the following integrals.

$$\int_0^1 (1-x)x^k dx = \frac{1}{k+1} - \frac{1}{k+2}, \quad \int_0^1 (1-x)^2 x^k dx = \frac{1}{k+1} - \frac{2}{k+2} + \frac{1}{k+3}$$

A MATLAB code that evaluates the matrices and gives the eigenvalues is:

```
% problem 2c
clear A B lambda
mmax = 2
for i=1:mmax
   for j=1:mmax
      A(j,i)=i*(i-1)*(1/(i+j-1)-1/(i+j))-i*(i+1)*(1/(i+j)-1/(i+j+1));
      B(j,i)=-(1/(j+i+1)-2/(j+i+2)+1/(j+i+3));
   end
end
A
B
lambda=eig(A,B)
```

Results are as follows.

N	λ1	λ2	λ3	λ4
1	10			
2	10	42		
3	9.8697	42	102.13	
4	9.8697	39.5	102.13	200.5
exact	9.8696044	39.4784176	88.8264396	157.9136704

Errors and error bounds are:

n	error1	bound1	error2	bound2	error3	bound3	error4	bound4
2	0.130	2.31	2.52	171.5				
3	1.45e-4	0.334	2.52	25.7	13.3	369		
4	1.45e-4	0.080	0.0231	5.14	13.3	88.8	42.6	672

Part d. in preparation

Part e. Following the instructions in the problem statement, choose $c = 1$, $a = f = e_a = 1$, $\alpha = \beta = \gamma = 0$, $d_a = 1$. The results are shown below. As expected, the values are above the exact solution, the errors increase for the higher eigenvalues, and the errors decrease as the number of elements increases. The eigenvalues also become more accurate as one transitions from Lagrange, linear to Lagrange, quadratic, to Hermite, cubic. This is also expected since there are more degrees of freedom in this transition.

Lagrange, linear

n	m=4(extra coarse)	m=7 (courser)	m=9 (course)	exact
1	10.20	9.997	9.951	9.8696044
2	44.89	41.55	40.79	39.47842
3		99.49	95.58	88.82644
4		192.0	179.6	157.9137
5		686.5	300.0	246.7401

Lagrange, quadratic

n	m=4(extra coarse)	m=7 (courser)	m=9 (course)	exact
1	9.8717	9.8699	9.8697	9.8696044
2	39.6050	39.4986	39.4868	39.47842
3	90.1490	89.0484	88.9195	88.82644
4	164.46	159.10	158.42	157.9137
5	1324.51	251.01	248.60	246.7401

Hermite, cubic

n	m=2(extremely coarse)	m=4 (extra course)	m=7 (courser)	exact
1	9.869912	9.869622	9.869606	9.8696044
2	39.5204	39.4817	39.47867	39.47842
3	90	88.8834	88.8317	88.82644
4	378	158.31	157.955	157.9137
5		250	246.933	246.7401