The following problem accompanies the book, Method of Weighted Residuals and Variational Principles, by Bruce A. Finlayson, a SIAM Classic reprinted in 2014. The original version was printed by Academic Press in 1972. See www.ChemEComp.com/MWR. Order the book from the Society of Industrial and Applied Mathematics, www.SIAM.org. The problems and solutions refer to equations and references in that book.

Problem 3.

Part a. The variational integral is

$$\lambda[y] = \frac{\int_0^1 \left(\frac{dy}{dx}\right)^2 dx}{\int_0^1 y^2\, dx}$$

Using

$$\left.\frac{dy}{dx}\right|_i = \frac{y_{i+1} - y_i}{\Delta x}$$

gives

$$\lambda[y] = \frac{\sum_{i=0}^{n+1} \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2 \Delta x}{\sum_0^{n+1} y_i^2 \Delta x}$$

Differentiating this with respect to y_j gives

$$2\frac{y_j - y_{j-1} - y_{j+1} + y_j}{\Delta x^2}\Delta x - 2\Delta x \lambda y_j = 0$$

The end points are $j = 0$ and $j = n+1$. For points 2 through n -1 we get

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{\Delta x^2} + \lambda y_j = 0$$

For the remaining points we get

$$\frac{y_2 - 2y_1 + y_0}{\Delta x^2} + \lambda y_1 = 0 \text{ and } \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2} + \lambda y_n = 0$$

and $y_0 = y_{n+1} = 0$. After rearrangement we want the eigenvalues of A_{ji} where $A_{ii} = -2, \quad A_{i,i-1} = A_{i,i+1} = 1$ and the eigenvalues (found in MATLAB using lambda = eig(A)) are values of $-\lambda \Delta x^2$.

Part b. A MATLAB code that evaluates the matrices and gives the eigenvalues is:

```
%eigenvalue using finite difference method
clear A lambda
nmax = 5
for i=2:nmax-1
   A(i,i)=-2;
   A(i,i+1) = 1;
   A(i,i-1) = 1;
end
A(1,1)=-2;
A(1,2)=1;
A(nmax,nmax)=-2;
A(nmax,nmax-1)=1;
A
lambda=eig(A)
delx2=(nmax+1)^2;
lambda=-lambda*delx2
```

Results are as follows.

n	3	4	5	8	10	exact
λ1	9.3726	9.5492	9.6462	9.7698	9.8217	9.8696044
λ2	32.0000	34.5492	36.0000	38.4166	38.7159	39.47842
λ3	54.6274	65.4508	72.0000	83.5237	85.0034	88.82644
λ4		90.4508	108.000	133.869	145.994	157.9137

Errors are:

n	3	4	5	8	10	exact
λ1	-0.497	-0.320	-0.223	-0.172	-0.0479	9.8696044
λ2	-7.48	-4.93	-3.48	-1.06	-0.763	39.47842
λ3	-34.2	-23.4	-16.8	-5.30	-3.82	88.82644
λ4		-67.5	-49.9	-24.0	-11.9	157.9137

For the finite difference method, the errors are much larger, they happen to be a lower bound, and the convergence with *n* is much slower. For *n* = 20, 30, and 40, the first and second eigenvalues are: 9.8512, 9.8612, 9.8648; 39.1848, 39.3435, 39.4012.