The following problem accompanies the book, Method of Weighted Residuals and Variational Principles, by Bruce A. Finlayson, a SIAM Classic reprinted in 2014. The original version was printed by Academic Press in 1972. See www.ChemEComp.com/MWR. Order the book from the Society of Industrial and Applied Mathematics, www.SIAM.org. The problems and solutions refer to equations and references in that book.

Problem 5. Part a. Using the expansion function, Eq. (11.87)

$$u_n(x,t) = \sum_{k=1}^{n} C_{nk}(t) \sin kx$$

one has

$$\frac{\partial u_n}{\partial t}(x,t) = \sum_{k=1}^{n} \frac{dC_{nk}}{dt} \sin kx, \quad \frac{\partial^2 u_n}{\partial x^2}(x,t) = -k^2 \sum_{k=1}^{n} C_{nk}(t) \sin kx$$

The residual is then

$$-\sum_{k=1}^{n} k^2 C_{nk}(t) \sin kx - \sum_{k=1}^{n} \frac{dC_{nk}}{dt} \sin kx - \sum_{k=1}^{n} C_{nk}(t) \sin kx = -\sum_{j=1}^{5} \frac{\sin jx}{10^{j-1}}$$

Making the residual orthogonal to the sine functions (Eq. 11.88) gives

$$-k^2 C_{nk}(t) \frac{\pi}{2} - \frac{dC_{nk}}{dt} \frac{\pi}{2} - C_{nk}(t) \frac{\pi}{2} = -\frac{1}{10^{k-1}} \frac{\pi}{2}$$

or

$$\frac{dC_{nk}}{dt} + (k^2+1) C_{nk}(t) - \frac{1}{10^{k-1}} = 0, \; C_{nk}(0) = 0$$

The solution can be found using an integrating factor, $e^{(k*k+1)t}$.

$$\frac{d}{dt}\left[e^{(k^2+1)t} C_{nk}(t)\right] = e^{(k^2+1)t} \left[\frac{dC_{nk}}{dt} + (k^2+1) C_{nk}(t)\right]$$

The differential equation can then be written in the form

$$\frac{d}{dt}\left[e^{(k^2+1)t} C_{nk}(t)\right] = e^{(k^2+1)t} \frac{1}{10^{k-1}}$$

Integrating once with respect to t gives:

$$e^{(k^2+1)t}C_{nk}(t) = \frac{1}{10^{k-1}}\int_0^t e^{(k^2+1)\tau}d\tau = \frac{1}{10^{k-1}}\frac{1}{k^2+1}\left[e^{(k^2+1)t}-1\right]$$

The solution is thus

$$C_{nk}(t) = \frac{1}{10^{k-1}}\frac{1}{k^2+1}\left[1-e^{-(k^2+1)t}\right], \quad \frac{dC_{nk}}{dt} = \frac{1}{10^{k-1}}e^{-(k^2+1)t}, \quad u(x,t) = \sum_{k=1}^{n} C_{nk}(t)\sin kx$$

Part b. Next calculate the residual and square it.

$$Lu_n - f = -k^2\sum_{k=1}^{n} C_{nk}(t)\sin kx - \sum_{k=1}^{n}\frac{dC_{nk}}{dt}\sin kx - \sum_{k=1}^{n} C_{nk}(t)\sin kx + \sum_{j=1}^{5}\frac{\sin jx}{10^{j-1}}$$

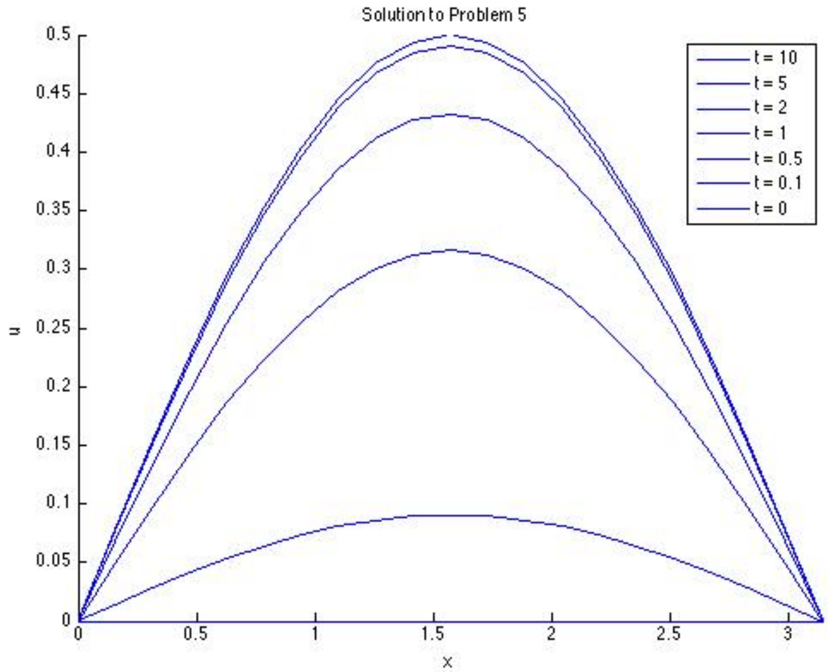
When the integration takes place, the sine functions are orthogonal, so one gets

$$\int_0^{\pi}(Lu_n - f)^2 dx = \left(\frac{\pi}{2}\right)\left[\sum_{k=1}^{n}(k^2+1)^2 C_{nk}^2 + \sum_{k=1}^{n}\left(\frac{dC_{nk}}{dt}\right)^2\right] + \left(\frac{\pi}{2}\right)\left[\sum_{j=1}^{5}\left(\frac{1}{10^{j-1}}\right)\right]^2$$
$$+\left(\frac{\pi}{2}\right)\left[2\sum_{k=1}^{n}(k^2+1)C_{nk}\frac{dC_{nk}}{dt} - 2\sum_{k=1}^{n}(k^2+1)C_{nk}\sum_{j=1}^{5}\left(\frac{1}{10^{j-1}}\right) - \sum_{k=1}^{n}\frac{dC_{nk}}{dt}\sum_{j=1}^{5}\left(\frac{1}{10^{j-1}}\right)\right]$$

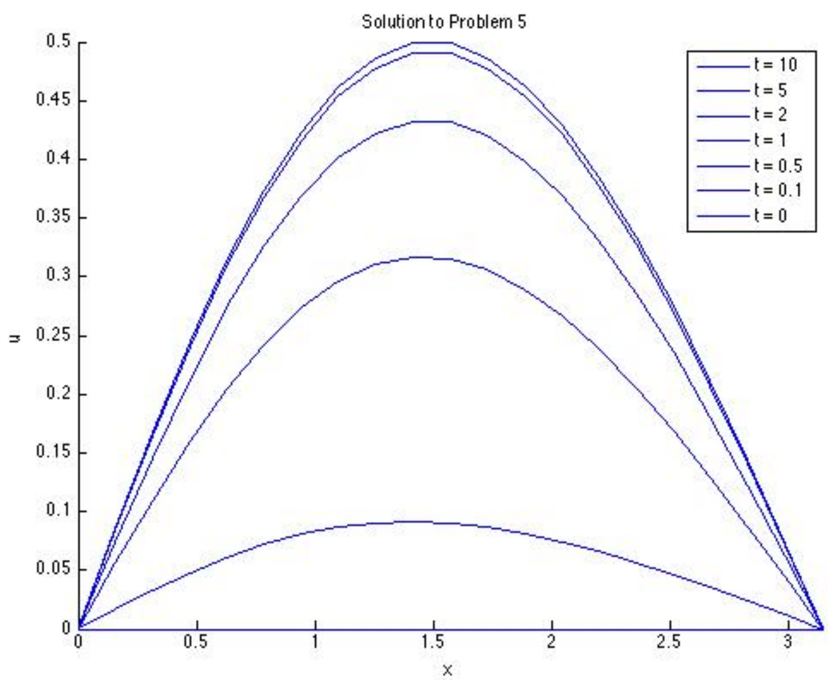
The error bounds are:

n	Residual at π/2	sqRes (same at all t)
1	−0.0099	0.0159
2	−0.0099	0.159e-3
3	1e-4	0.159e-5
4	1e-4	0.159e-7
5	0	1e-15
6	0	0

The sqRes is the same at all times because it is essentially from the (1/10^(j-1)) terms not included when n<5. The solution is exact when n = 5 or higher. Plots of the solution are given below. The solutions are very close, but not identical.



Solution for *n* = 1



Solution for *n* = 5

The error bound does change with time, as indicated in the table. The values of error bound are the square root of Eq. (11.89). The error does grow as times proceeds (Green indicated that), but it decreases as the number of terms is increased.

n	t = 0	t = 0.1	t = 0.5	t = 1	t = 2	t = 5	t = 10
1	0	0.0355	0.1775	0.3549	0.7098	1.7746	3.5491
2	0	0.0112	0.0561	0.1122	0.2245	0.5612	1.1223
3	0	0.0035	0.0177	0.0355	0.0710	0.1775	0.3549
4	0	0.0011	0.0056	0.0112	0.0224	0.0560	0.1120
5	0	0	0	0	0	0	0