The following problem accompanies the book, Method of Weighted Residuals and Variational Principles, by Bruce A. Finlayson, a SIAM Classic reprinted in 2014. The original version was printed by Academic Press in 1972. See www.ChemEComp.com/MWR. Order the book from the Society of Industrial and Applied Mathematics, www.SIAM.org. The problems and solutions refer to equations and references in that book.

Problem 6, Part a. Take the first variation:

$$\delta I = \int_A \left[-2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial \delta u}{\partial x}\right) - 2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial \delta u}{\partial y}\right) + 2\delta u \right] dA$$

Integrate by parts to get

$$\delta I = \int_A \left[-2\frac{\partial}{\partial x}\left(\delta u \frac{\partial u}{\partial x}\right) - 2\frac{\partial}{\partial y}\left(\delta u \frac{\partial u}{\partial y}\right)\right] dA + \int_A \left[2\delta u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\delta u \right] dA$$

Then using the divergence theorem we get

$$\delta I = \int_C \left[-2\delta u\, \mathbf{n} \bullet \nabla u \right] dc + 2\int_A \left[\delta u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 1\right)\right] dA$$

Since $\delta u = 0$ on the boundary, the correct Euler equation is derived.

Taking the second variation gives

$$\delta^2 I = \int_A \left[-2\left(\frac{\partial \delta u}{\partial x}\right)\left(\frac{\partial \delta u}{\partial x}\right) - 2\left(\frac{\partial \delta u}{\partial y}\right)\left(\frac{\partial \delta u}{\partial y}\right) - 2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial \delta^2 u}{\partial x}\right) - 2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 \delta u}{\partial y}\right) + 2\delta^2 u \right] dA$$

The last three terms are treated as before, giving zero for the exact solution, and leaving

$$\delta^2 I = -2\int_A \left[\left(\frac{\partial \delta u}{\partial x}\right)\left(\frac{\partial \delta u}{\partial x}\right) + \left(\frac{\partial \delta u}{\partial y}\right)\left(\frac{\partial \delta u}{\partial y}\right)\right] dA$$

This is always negative, so the variational principle is a maximum principle and the variational integral is a lower bound on the integral evaluated with the exact solution, Eq. (7.65).

Part b. The trial function is

$$u(x,y) = \left(\frac{1}{4} - x^2\right)\left(1 - y^2\right)\sum_{i=1}^{nx}\sum_{j=1}^{ny} a_{(i-1)*ny+j}\, x^{2i-2} y^{2j-2}$$

and the *x* and *y* derivatives are

$$\frac{\partial u}{\partial x} = \sum_{i=1}^{nx}\sum_{j=1}^{ny} a_{(i-1)*ny+j}\left(1 - y^2\right) y^{2j-2}\left[\frac{1}{4}(2i-2)x^{2i-3} - 2i\, x^{2i-1}\right]$$
$$\frac{\partial u}{\partial y} = \sum_{i=1}^{nx}\sum_{j=1}^{ny} a_{(i-1)*ny+j}\left(\frac{1}{4} - x^2\right) x^{2i-2}\left[(2j-2)y^{2j-3} - 2j\, y^{2j-1}\right]$$

Note that some terms drop out for certain *i* and *j*. If you are programming the solution, you need to omit the terms with negative powers. The variational method requires taking derivatives with respect to the parameters *a*.

$$\frac{\partial u}{\partial a_{(i-1)*ny+j}} = \left(\frac{1}{4} - x^2\right)\left(1 - y^2\right) x^{2i-2} y^{2j-2}$$
$$\frac{\partial}{\partial a_{(i-1)*ny+j}}\left(\frac{\partial u}{\partial x}\right) = (1 - y^2)\, y^{2j-2}\left[\frac{1}{4}(2i-2)x^{2i-3} - 2i\, x^{2i-1}\right]$$
$$\frac{\partial}{\partial a_{(i-1)*ny+j}}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{1}{4} - x^2\right) x^{2i-2}\left[(2j-2)y^{2j-3} - 2j\, y^{2j-1}\right]$$

First use only one term, *a*. The integrals are the square of the terms above. The three integrals take the values: $-a^2 \bullet 0.08889$, $-a^2 \bullet 0.02222$, and $2a \bullet 0.05556$. Thus, the value of *a* is 0.5. The value of the variational integral is

$$I = -\left(\frac{1}{2}\right)^2 \left(0.08889 + .02222\right) + 2\left(\frac{1}{2}\right) 0.05556 = 0.02778$$

If one integrates the variational integral by parts following the same steps used when finding the first variation, we get

$$I(u) = \int_A \left[-\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 + 2u\right] dA = \int_A \left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 1\right)\right] dA + \int_A u\, dA + \int_C \left[-2u\,\mathbf{n} \bullet \nabla u\right] dc$$

The first term is zero for the exact solution, and the boundary term is zero because of the boundary conditions. Thus, the integral is the value of the flow rate [note the nomenclature error is Eq. (7.65)]. To get the flow rate (or *I)* over the entire domain, multiply by 4.

$$\text{flow rate} = \int_A u\,dA, \quad \text{average velocity } \langle u \rangle = \frac{1}{A}\int_A u\,dA = \frac{I(u)}{A}$$

The formula for *f*Re is Eq. (4.11).

$$f\text{Re}' = K', K' = \frac{2A}{Cd'}\frac{1}{\langle u \rangle} \quad \text{and} \quad f\text{Re} = \frac{K'd}{d'}$$

The parameter *d* ‘ is the dimension used to make the equations dimensionless (here = 1), the *d* is the 4 x the hydraulic radius = 4A/C (4/3), and the *A* and *C* are the cross sectional area and circumference, here 2 and 6. Thus,

$$\langle u \rangle = \frac{I}{A} = \frac{4 \bullet 0.02778}{4 \bullet 0.5} = 0.05555,\ K' = \frac{2A}{C}\frac{1}{\langle u \rangle} = \frac{2*4*0.5}{6*0.05555} = 12 \quad \text{and} \quad f\,\text{Re} = \frac{K'(4/3)}{1} = 16$$

Check that the average velocity is less than the true value. The exact value is 0.05717, so the approximate value is about 3% low. The exact value of *f*Re is 15.55, so the error is about 3%, of course. Thus, the first term in the approximation gives excellent results. The exact values are listed in Table 4.2, except that they are there for an aspect ratio of 0.4 and 0.6, but not 0.5. Doing a linear interpolation gives 15.65. The more precise value is determined in part c.

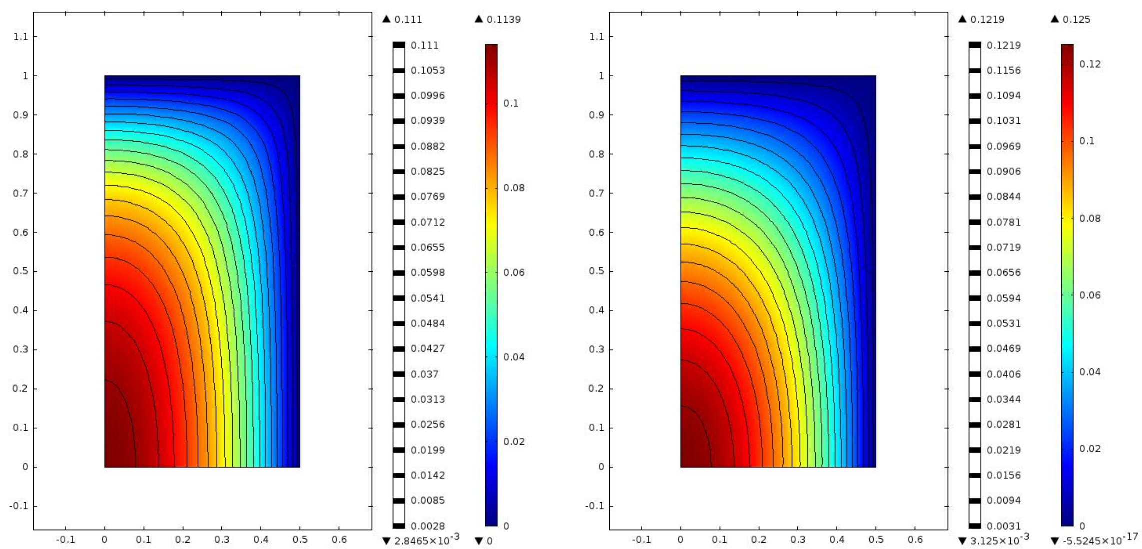
Calculations for higher values of *nx* and *ny* are given in the table as computed by the MATLAB program problem_6.m. Note that the value of *I* goes up as an additional degree of freedom is introduced (due to the maximum principle). Of course, when one degree of freedom is reduced and another increased that doesn't necessarily happen (see *nx* =2 and *ny* = 4 and *nx* =3 and *ny* = 3). The value of *I*(*u*) increases as the approximation becomes better. It is a lower bound to the exact value of the variational integral. The estimated error is found using the last value for *nx* =6 and *ny* = 12.

nx	*ny*	Integral of 2*u* on quarter domain	*I* on quarter domain	Est. |Error| on total domain
1	1	0.055555	0.02777778	8.1e-4
1	2	0.057060	0.02853022	5.5e-5
2	2	0.057141	0.02857047	1.5e-5
2	3	0.057168	0.02858377	1.4e-6
2	4	0.057169	0.02858434	8.8e-7
3	3	0.057168	0.02854239	9.7e-7
3	4	0.057170	0.02858505	1.6e-7
3	5	0.057170	0.02858514	7.2e-8
3	6	0.057170	0.02858515	6.3e-8
6	12	0.057170	0.02858521	–

Part c. The program Comsol Multiphysics was used to solve the partial differential equation with the finite element method. One option is Poisson's equation (in the Mathematics section). Define a rectangle going from x = 0 to 0.5, y = 0 to 1. The default values for Poisson's equation are appropriate here, namely c = 1, f = 1. Use Dirichlet boundary conditions on the top and right-side boundary, with u = 0. The boundary conditions on the left and bottom are normal derivative zero (zero flux). Choose the discretization as Hermite cubic polynomials, which are continuous and have continuous first derivatives. The results are shown in the table.

Mesh description	dof	<u>	I(u) (total domain)	u(0,0)
Extremely coarse	86	.057163	.114326	.113862
Extra coarse	122	.057169	.114337	.113870
Coarser	254	.057170	.114340	.113871
Coarse	368	.057170	.114341	.113872
Normal	857	.057170	.114341	.113872

While this does not provide error bounds (see the discussion on page 360), the rate of convergence is useful. Examine Schultz (1969c) to see if the theorems there can provide an error bounds. A plot of the solution is shown below, along with a plot of the first approximation derived in part b. The first approximation is slightly in error since the formula in the y direction only has one term and it is twice as long as the distance in the x direction.



Left: Comsol solution for extremely coarse mesh (86 dof) and right: approximate solution with one unknown (part b)

It is of interest to compare the estimated errors in part b with those in part c. The figures shows that the global polynomial approximation reduces the error with only a few terms, while the finite element method using cubic Hermit polynomials requires many more terms to achieve the same accuracy.